

A CLASS OF TWO-STEP TVD MACCORMACK TYPE NUMERICAL SCHEME FOR MHD EQUATIONS^{*}

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Abstract

In this paper, a new numerical scheme of Total Variation Diminishing (TVD) MacCormack type for MagnetoHydroDynamic (MHD) equations is proposed by taking into account of the characteristics such as convergence, stability, resolution. This new scheme is established by solving the MHD equations with a TVD modified MacCormack scheme for the purpose of developing a scheme of quick convergence as well as of TVD property. To show the validation, simplicity and practicability of the scheme for modelling MHD problems, a self-similar Cauchy problem with the discontinuous initial data consisting of constant states, and the collision of two fast MHD shocks, and two-dimensional Orszag and Tang's MHD vortex problem are discussed with the numerical results conforming to the existing results obtained by the Roe type TVD, the high-order Godunov scheme, and Weighted Essentially Non-Oscillatory (WENO) scheme. The numerical tests show that this two-step TVD MacCormack numerical scheme for MHD system is of robust operation in the presence of very strong waves, thin shock fronts, thin contact and slip surface discontinuities.

Key words Two step TVD MacCormack scheme; MHD numerical simulation

1 Introduction

Ideal magnetohydrodynamics deals with a coupling between Euler's equations and Maxwell's equations. It can be used as a simplified model for the simulation of the space weather, which results from the interaction of the solar wind with the magnetosphere of

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the earth. As a basic model of describing space plasma, MHD is a non-strictly hyperbolic system composed of eight equations coupled by fluid dynamics and magnetic induction. Differing from the usual fluid dynamic equations with the unique sound wave mode, MHD has multiple modes of sound wave, Alfvén wave, fast magnetosonic and slow magnetosonic wave, which make the eigenvalues and eigenvectors of MHD equations so complex that many modern high resolution numerical schemes used in simulating supersonic flow problems of computational fluid mechanics can not be transplanted directly to the numerical study of supersonic, super-Alfvénic solar wind flow. Along with the recent advance of space weather, great attention has been paid to the development of quick convergence and high resolution numerical schemes of MHD system for the purpose of solar wind storm simulation or establishing numerical prediction methods of space weather. For the numerical methods currently used in solar wind simulation, we can refer to Feng *et al*^[3]. We have seen great efforts made in theory of MHD and application of solar wind simulation there.

It is noticed that for routine second-order, finite difference schemes of hyperbolic equations, oscillations of a central scheme and a bias (upwind) scheme always take place at opposite sides of a discontinuity. So spurious oscillations should be corrected or reduced to as little as possible via some selective employments of the second-order schemes. It is also known that a multi-step high-order scheme shares the merits of not using the Jacobian matrix. Based partially on this observation, a TVD Lax-Friedrichs scheme is established by using modified Lax-Friedrichs scheme for the fluid equations of the MHD system and the well-known MacCormack II for the magnetic induction equations in spherical coordinates^{[3],[4]}. The modified Lax-Friedrichs scheme consist of two steps of predictor and corrector steps and is of second order accuracy both in time and space. Compared to those of routine TVD and Godunov type, the TVD Lax-Friedrichs scheme need not use the calculation of eigenvalues and eigenvectors associated to the Jacobian matrices in every time step and thus reduce a lot of work. Numerical results from numerical solar wind modelling and magnetic reconnection in interplanetary space have shown the good merit of this scheme^{[3],[10],[4]}, which further motivates us to explore two-step numerical method for MHD system. Here, a two-step TVD scheme based on the well-known MacCormack scheme is proposed after giving some remarks about MHD system. Then we exhibit some numerical results for nonlinear problems particular to MHD system.

2 Some Remarks on MHD System

The induction equation^[9]

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times \mathbf{E} \quad (1)$$

is combined with the equations of gas dynamics to describe the behavior of compressible electrically conducting fluids subject to magnetic fields. For nonrelativistic fluids, where

Maxwell's displacement current may be neglected, the combined system is referred to as the magnetohydrodynamic (MHD) equations. The compressible ideal (inviscid and perfectly conducting) MHD equations may be written as a hyperbolic system of conservation laws in the form^[14]

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \mathbf{B} \\ U \end{pmatrix} + \left\{ \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \mathbf{v} + \left(p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right) \mathbf{I} - \mathbf{B} \mathbf{B} \\ \mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v} \\ \left(U + p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right) \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \end{pmatrix} \right\}^T = 0, \quad (2)$$

where $\mathbf{v}$ is the fluid velocity, ρ the density, and p the pressure. The electric field $\mathbf{E}$ has been eliminated using the perfectly conducting condition $\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0$ appropriate for highly conductive media such as space and astrophysical plasmas. The symbol $\mathbf{I}$ denotes the 3×3 identity tensor with components δ_{ij} . A factor of $1/\sqrt{\mu_0}$ has been absorbed into the definition of $\mathbf{B}$. The total energy density U comprising the internal, kinetic, and magnetic energy is given by

$$U = \frac{1}{2} \rho v^2 + \frac{p}{\gamma - 1} + \frac{1}{2} \mathbf{B}^2 \quad (3)$$

for a perfect gas with ratio of specific heats γ .

The evolution in time of a magnetic field $\mathbf{B}$ is determined by an electric field $\mathbf{E}$ through the induction equation (1), one of Maxwell's equations. The magnetic field must also satisfy $\nabla \cdot \mathbf{B} = 0$. This constraint expresses the absence of magnetic monopoles, which have never been observed experimentally. Since (1) implies $\partial_t(\nabla \cdot \mathbf{B}) = 0$, this constraint is often treated as an initial condition, which will be preserved under subsequent evolution.

The MHD system is much more complex than the usual fluid dynamics system. On one hand, there are four kinds of discontinuities which may exist in a general system of ideal MHD: contact discontinuities, fast shocks, slow shocks, and rotational discontinuities. In the special case with vanishing B_x , there are tangential discontinuities and magnetosonic shocks. A detailed description of MHD discontinuities may be found in many standard texts (for example, Jeffrey and Taniuti^[5], Kulikovskiy and Lyubimov^[6]). On the other hand, the one-dimensional MHD equations are degenerate. If the variables in (2) are functions of one coordinate n only, the evolution equation for B_n becomes simply $\partial_t B_n = 0$. In a sense, this is consistent with the one-dimensional form of the constraint $\nabla \cdot \mathbf{B} = 0$, which simplifies to $\partial_n B_n = 0$. Thus the magnetic field B_n normal to the discontinuity should be constant in the initial conditions and will then remain constant. In a finite volume discretisation, $\nabla \cdot \mathbf{B} = 0$ implies that the signed sum of the jumps in the normal component of $\mathbf{B}$ across a cell's boundary should vanish. Thus the initial data for the one-dimensional Riemann problems, if used to compute fluxes in a multidimensional scheme, will generally contain jumps in B_n comparable to the jumps in the tangential components $\mathbf{n} \times \mathbf{B}$.

A particular and challenging problem in the numerical treatment of the ideal MHD equations concerns the physical constraint that magnetic monopoles do not exist ($\nabla \cdot \mathbf{B} = 0$).

This condition is not always exactly preserved after the discretization of the equations. Several strategies have been used to deal with the $\nabla \cdot \mathbf{B} = 0$ constraint. One approach is to use the solution of the reduced seven-wave Riemann problem^[19], using some average value for B_n on both sides, to update the seven variables other than B_n , followed by a separate step which updates B_n so as to preserve $\nabla \cdot \mathbf{B} = 0$ ^{[2],[15]}. Another approach, pioneered by Powell *et al*^[14], adds terms proportional to $\nabla \cdot \mathbf{B}$ to the system (2) to make the one-dimensional Riemann problem nondegenerate. It is worth emphasizing that although Powell's approach maintains $\nabla \cdot \mathbf{B} \sim 0$ to truncation error in a multidimensional sense, in the one-dimensional Riemann problems $\nabla \cdot \mathbf{B}$ is comparable to $\nabla \times \mathbf{B}$. The third one has been proposed by Toth^[18] in order to avoid the non conservative formulation of the source term in the Powell approach. Instead of the source term, Toth^[18] has proposed a temporal and space averaging for the update of the magnetic field directly using the induction equation. Numerical experiments comparing various schemes to show the ability of keeping the $\nabla \cdot \mathbf{B}$ constraint have been performed by Toth^[18].

Very recently, Hans^[26] proposed that upwind constrained transport on unstructured grids can effectively control the $\nabla \cdot \mathbf{B} = 0$ constraint and pointed out that this kind of scheme can be applied to any hyperbolic system with divergence-free fields.

It should be noticed that numerical schemes in non-conservative form of MHD system can also maintain the $\nabla \cdot \mathbf{B} = 0$ constraint to a tolerable error^[3].

According to Godunov^[20], Powell *et al*^[14] proposed a modified form

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \mathbf{B} \\ U \end{pmatrix} + \left\{ \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \mathbf{v} + \left(p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right) \mathbf{I} - \mathbf{B} \mathbf{B} \\ \mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v} \\ \left(U + p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B} \right) \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \end{pmatrix}^T \right\} = -\nabla \cdot \mathbf{B} \begin{bmatrix} 0 \\ \mathbf{B} \\ \mathbf{v} \\ \mathbf{v} \cdot \mathbf{B} \end{bmatrix}, \quad (4)$$

in which source terms proportional to $\nabla \cdot \mathbf{B}$ have been added to the momentum and induction and energy equations. This system was constructed by modifying the coefficient matrix in the linearized Riemann problem to include an eighth wave corresponding to passive advection of jumps in B_n with the fluid speed v_n . This is the only possibility that leaves the system invariant under the Galilean transformations $\mathbf{x} \rightarrow \mathbf{x} + \mathbf{v}_0 t$ and $\mathbf{v} \rightarrow \mathbf{v} + \mathbf{v}_0$, while $\mathbf{B}$ and t remain unchanged.

Janhunen^[13] found that the solution of the Riemann problem for Powell's system (4) for left and right states with positive fluid pressures may contain an unphysical intermediate state with negative fluid pressure. It is known that this so-called lack of positivity is a particular problem for astrophysical applications, where the contribution to the total energy from the fluid pressure is often small compared with the magnetic and possibly the kinetic energy. Thus computing the fluid pressure p from the conserved quantities $\rho, \rho \mathbf{v}, \mathbf{B}$, and U often involves the difference between two nearly equal terms. Janhunen^[13] found that positivity, as well as local energy and momentum conservation, could be regained by

discarding the source terms in the energy and momentum equations, so that (4) becomes

$$\frac{\partial}{\partial t} \begin{pmatrix} \rho \\ \rho \mathbf{v} \\ \mathbf{B} \\ U \end{pmatrix} + \left\{ \nabla \cdot \begin{pmatrix} \rho \mathbf{v} \\ \rho \mathbf{v} \mathbf{v} + \left(p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B}\right) \mathbf{I} - \mathbf{B} \mathbf{B} \\ \mathbf{v} \mathbf{B} - \mathbf{B} \mathbf{v} \\ \left(U + p + \frac{1}{2} \mathbf{B} \cdot \mathbf{B}\right) \mathbf{v} - (\mathbf{v} \cdot \mathbf{B}) \mathbf{B} \end{pmatrix}^T \right\}^T = -\nabla \cdot \mathbf{B} \begin{bmatrix} 0 \\ 0 \\ \mathbf{v} \\ 0 \end{bmatrix}. \quad (5)$$

Dellar^[12] pointed out that Powell's system (4) fails to conserve momentum and energy.

Thus, taking account of the above consideration, we will employ Janhunen's system (5) for our implementation of the numerical scheme.

3 Numerical Scheme Formulation

Since the birth of Harten's concept of total variation diminishing (TVD)^[22], a new era of high order and high-resolution schemes for hyperbolic conservation laws has been initiated. TVD schemes are a kind of monotonicity preserving schemes by adding and limiting a modified, high-order anti-diffusion term to a low-order (e.g., first-order upwind) numerical flux. They produce a high-resolution of discontinuities while retaining high-order accuracy in smooth regions. Following the TVD concept, many numerical schemes have been designed^{[14],[15],[17],[21],[25],[27]}. Almost all these modern shock capturing methods are based on the local characteristic decomposition of waves. The fluxes are first separated into each family of waves. Then they are split into positive and negative parts by either global or local Lax Friedrichs flux splitting. Subsequently, the fluxes at the cell boundaries are constructed by the TVD or WENO approximation. Because of the wave decomposition procedure, considerable work is required to evaluate the associated eigensystem. There are seven waves in the MHD system. In addition to the entropy wave, which propagates with the fluid speed, $\mathbf{v}$, there are three wave modes, which can propagate with two $\mathbf{v} \pm c$ speeds, where c denotes the characteristic speed of the mode. According to the magnitude of the wave speeds, these three modes are called fast, intermediate (Alfven), and slow waves. The fast and slow waves are compressive, while the intermediate wave is not. Depending on the direction and the magnitude of the magnetic field, these wave speeds may coincide. Thus the MHD equations form a non-strictly hyperbolic system. In spite of this, a complete set of eigenvectors can be found and, with proper normalizations, they are well-defined^{[15],[19],[27]}. Characteristic decomposition of waves depend on the exact solution of the MHD Riemann problem, which may be not unique^[6], in fact, is too multivariant to be used in regular calculations. The main efforts in these modern upwind methods is the development of a viable Riemann solver based on the knowledge about the eigensystem of the flow equations. Moreover, for computational efficiency, further approximation of the Riemann solvers employed is often developed. However, unlike the gas dynamics equations, to which analytical solution in

one spatial dimension is well-known, available analytical solutions to the MHD waves have been scant. Therefore, the development of approximate Riemann solvers for MHD equations involves many theoretical uncertainties. When extending to plasma flows in multiple spatial dimensions, the common practices in upwind methods are based on directional splitting, i.e., waves propagating in a multi-dimensional space are approximated by superposition of one-dimensional Riemann solutions from each of the orthogonal directions. While the practice is standard in calculating gas dynamics equations, direct extension of this treatment for complex MHD equations need further study. One may question the validity of using these scheme for MHD simulations.

The schemes based on the approximate Riemann solver are very time-consuming since these kinds of schemes require the splitting of the associated Jacobian matrices and the calculation of the eigenvalues and eigenvectors at every step. Thus, to avoid the use of the Riemann solver, we need to explore some simple schemes not based on the use of complete set of eigenvectors. Here we will pay our attention to the modification of the well known MacCormack scheme because of its simplicity. Based on the flux splitting technique, a MacCormack TVD scheme has long been used in hydraulic dynamics^{[7],[16]}, especially for the computation of Saint-Venant equations, in the context of coastal hydrodynamics. A modification to the well known Beam and Warming implicit scheme based on the concept of conservative splitting of flux through an approximate Jacobian is proposed and applied to one-dimensional unsteady free-surface flows^[8]. TVD MacCormack and TVD Warming-Beam schemes have also been introduced in Tan^[24]. Yu and Liu^[11] present a two-step, component-wise TVD scheme for nonlinear, hyperbolic conservation laws, which is obtained by combining the schemes of Mac Cormack and Warming-Beam. The latter scheme does not necessitate the characteristic decompositions of the usual TVD schemes by employing component-wise limiting. In what follows we will see how to realize two-step TVD MacCormack scheme for MHD system. To begin with, consider the nonlinear scalar equation

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} f(u) = 0. \quad (6)$$

It is useful to note that

$$f(u) = f^+(u) + f^-(u), \quad a^+ = \frac{df^+(u)}{du} \geq 0, \quad a^- = \frac{df^-(u)}{du} \leq 0. \quad (7)$$

For left-moving waves, i.e., $f^+ = 0$ in (7), we use the right upwind predictor of the MacCormack scheme

$$\tilde{u}_j = u_j^n - \lambda \delta^+ f_j^n, \quad (8)$$

then the corresponding corrector has the following form,

$$u_j^{n+1} = \frac{1}{2}(u_j^n + \tilde{u}_j) - \frac{1}{2}\lambda \delta^- \tilde{f}_j, \quad (9)$$

where $\delta^+(\cdot)_j = (\cdot)_{j+1} - (\cdot)_j$, $\delta^-(\cdot)_j = (\cdot)_j - (\cdot)_{j-1}$, $\lambda = \frac{\Delta t}{\Delta x}$.

For right-moving waves, i.e., $f^- = 0$ in (7), we use the left upwind predictor based on MacCormack formulation

$$\tilde{u}_j = u_j^n - \lambda \delta^- f_j^n \quad (10)$$

with the following corrector

$$u_j^{n+1} = \frac{1}{2}(u_j^n + \tilde{u}_j) - \frac{1}{2}\lambda \delta^+ \tilde{f}_j. \quad (11)$$

When both left-moving and right-moving waves appear, we need a flux splitting of (7). Substitute the predictors of (8) and (10) for $\tilde{u}_j$ in their relevant corrector steps (9) and (11), and rewrite the above schemes in the conservative form to get an extremely simple form of the numerical fluxes of the schemes for a general nonlinear scalar equation

$$\begin{aligned} \tilde{u}_j &= u_j^n - \lambda(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}), \\ u_j^{n+1} &= u_j^n - \lambda(\hat{H}_{j+\frac{1}{2}} - \hat{H}_{j-\frac{1}{2}}), \\ \hat{H}_{j+\frac{1}{2}} &= \hat{f}_j + \frac{1}{2}[\hat{g}_{j+\frac{1}{2}}^+ - \hat{g}_{j+\frac{1}{2}}^-]. \end{aligned} \quad (12)$$

where

$$\hat{g}_{j+\frac{1}{2}}^+ = \tilde{f}_{j+1}^+ - f_j^+, \hat{g}_{j+\frac{1}{2}}^- = f_{j+1}^- - \tilde{f}_j^-. \quad (13)$$

Following the flux limiting procedure of the modified Lax-Wendroff scheme^[21] by adding a pure flux limiter to the second-order antidiffusive term of (12), we get a modified second-order predictor-corrector scheme,

$$\tilde{u}_j = u_j^n - \lambda(\hat{f}_{j+\frac{1}{2}} - \hat{f}_{j-\frac{1}{2}}), \quad (14)$$

$$u_j^{n+1} = u_j^n - \lambda(\hat{\mathcal{H}}_{j+\frac{1}{2}} - \hat{\mathcal{H}}_{j-\frac{1}{2}}), \quad (15)$$

$$\hat{\mathcal{H}}_{j+\frac{1}{2}} = \hat{f}_j + \frac{1}{2}[\phi(r_{j+\frac{1}{2}}^+) \hat{g}_{j+\frac{1}{2}}^+ - \phi(r_{j+\frac{1}{2}}^-) \hat{g}_{j+\frac{1}{2}}^-]. \quad (16)$$

where

$$\hat{f}_j = f_j^+ + f_{j+1}^-, \quad r_{j+\frac{1}{2}}^+ = \hat{g}_{j-\frac{1}{2}}^+ / \hat{g}_{j+\frac{1}{2}}^+, \quad r_{j+\frac{1}{2}}^- = \hat{g}_{j+\frac{3}{2}}^- / \hat{g}_{j+\frac{1}{2}}^-, \quad (17)$$

Here $\phi(r) = 1$ is obviously reduced to the usual MacCormack scheme.

For the modified MacCormack scheme (14) and (15), all the negative flux terms vanish in the linear case $u_t + au_x = 0$, which is absolutely TVD under the condition given by Sweby^[21]. But, for the nonlinear case, we may as well refer to the above scheme as a two-step TVD scheme whenever a pure flux limiter $\phi(r)$ bounded by the usual TVD constraint is applied to the modified MacCormack scheme; although it is impossible to be abstractly proved up to now. Here we choose a Roe type flux limiter

$$\phi(r) = \begin{cases} r, & |r| < 1; \\ 1, & |r| \geq 1. \end{cases}$$

We can refer to Fu and Ma^[23] for details about this topic.

The two-step TVD scheme does not need to use the set of eigenvalues of the Jacobian matrix, but an appropriate flux splitting is still required for its upwinding procedure. As in Ref.[3], here we choose the simplest and inexpensive global Lax-Friedrichs flux splitting

$$f_j^\pm(u) = \frac{1}{2}(f(u_j) \pm \alpha u_j), \quad (18)$$

where

$$\alpha = \max_j |f'(u_j)|.$$

Generally speaking, for the scalar case, the TVD technique is elegant and amenable to rigorous analysis. However, when applied to systems of conservation laws, TVD schemes are often built upon field-by-field decompositions. One extends the scalar scheme to systems of conservation laws by applying it “scalarly” to each of the (appropriately linearized) characteristic variables^{[23],[24]}. For one-dimensional vector form, we just formally extend the scheme component-wisely.

For two or three dimensional systems, we extend the component wise TVD scheme by the method of fractional steps or time splitting or ADI. If system (5) in three dimensional Cartesian coordinates is written

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} + \frac{\partial G(U)}{\partial y} + \frac{\partial H(U)}{\partial z} = S, \quad (19)$$

the following form is suggested in the present paper. First, we can write (19) as three

$$\frac{\partial U}{\partial t} + \frac{\partial F(U)}{\partial x} = 0, \quad \frac{\partial U}{\partial t} + \frac{\partial G(U)}{\partial y} = 0, \quad \frac{\partial U}{\partial t} + \frac{\partial H(U)}{\partial z} = S. \quad (20)$$

We can implement the scheme (14)—(18) for the three equations in (20). If we denote the difference operator generated with (14)—(18) by $L_x(\Delta t)$, then the three equations have the following approximation

$$U_i^{n+1} = L_x(\Delta t)U_i^n, \quad U_j^{n+1} = L_y(\Delta t)U_j^n, \quad U_k^{n+1} = L_z(\Delta t)U_k^n.$$

Then the numerical scheme for (19) can be realized by

$$U_{i,j,k}^{n+1} = L_z\left(\frac{\Delta t}{2}\right)L_y\left(\frac{\Delta t}{2}\right)L_x(\Delta t)L_y\left(\frac{\Delta t}{2}\right)L_z\left(\frac{\Delta t}{2}\right)U_{i,j,k}^n, \quad (21)$$

and correspondingly we can choose the maximum of

$$\alpha_Q = |v_w| + c_Q^f = |v_w| + \frac{1}{\sqrt{2}} \sqrt{\frac{\gamma p + |\mathbf{B}|^2/\mu_0}{\rho}} + \sqrt{\left(\frac{\gamma p + |\mathbf{B}|^2/\mu_0}{\rho}\right)^2 - 4\frac{\gamma p B_w^2}{\mu_0 \rho^2}},$$

in place of (18) for the difference operators L_x, L_y, L_z , where c_Q^f ($Q = F, G, H$) is the fast-mode MHD wave, relative to the fluid in the $w = x, y, z$ directions. The CLF condition is chosen as

$$\Delta t \leq C \min \left\{ \frac{\Delta x}{\alpha_F^{\max}}, \frac{\Delta y}{\alpha_G^{\max}}, \frac{\Delta z}{\alpha_H^{\max}} \right\}.$$

where the maxima $\alpha_Q^{\max}$ is the maximal eigenvalue of $\partial Q / \partial U$, taken over all computational grid points, with $Q = F, G, H$ being fluxes and U independent variables, respectively. The Courant number C here can be taken to be less than 1 (0.5 or 0.8).

4 Numerical Results of One-Dimensional Shock Problems

In order to see the shock-capturing ability of the proposed TVD type MacCormack numerical scheme we here give two one-dimensional MHD shock problems.

As the first example we choose the same ideal MHD Riemann problem, as that of Brio and Wu^[15] with the initial data consisting of constant states at rest at $x = 0$. To the left of the discontinuity, $\rho_l = 1$, $(v_x)_l = 0$, $(v_y)_l = 0$, $p_l = 1$, $(B_y)_l = 1$; to the right, $\rho_r = 0.125$, $(v_x)_r = 0$, $(v_y)_r = 0$, $p_r = 0.1$, $(B_y)_r = -1$ with $B_x = 0.75$ and $\gamma = 2$. Because the MHD equations are nonconvex, compound waves (waves consisting of a shock followed by a rarefaction wave of the same family), which do not occur in hydrodynamics, can indeed exist in MHD. The coplanar Riemann problem presented here tests the ability of the scheme to represent shocks, rarefaction fans, and compound waves in MHD flows. Note that the magnetic field component B_y changes its sign, or rotates, to the angle π across the initial jump. In Ref.[15] different numerical schemes were tried. All of them give qualitatively the same picture of the flow, but the Roe's extension to MHD equations produces much sharper discontinuities with the absence of spurious oscillations. Figure 1 displays the self-similar distributions of the density ρ , the velocity components v_x and v_y , the magnetic field component B_y , the pressure p and the temperature T obtained for the coplanar problem. The number of grid points is 800 with $\Delta x = 1.0$ and $\Delta t = 0.2$ (CFL number 0.8). This solution is shown after 400 time steps. The initial discontinuity is located in the middle of computational domain. One can see fast rarefaction waves FR, a contact discontinuity CD, a compound wave CW, a slow rarefaction wave SS FR,

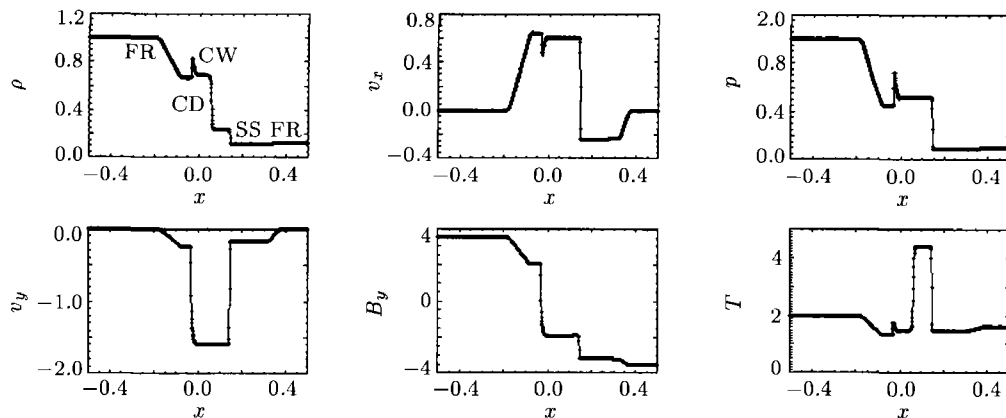


Fig.1 The self-similar distributions of MHD parameters at $t = 0.1$

a slow shock wave SS, and a slow compound wave CW. The results are in a good agreement with those obtained by the method of Brio and Wu^[15].

Another problem concerns the collision of two fast shocks with the initial condition for $(\rho, p, v_x, v_y, v_z, B_y, B_z)$ being $(3.877\,41, 75.149\,5, 7.685\,74, -0.0894\,647, -0.089\,464\,8, 7.821\,80, 7.821\,80)$ for $x < 0.025$, $(1, 0.1, 0, 0, 0, 2, 2)$ for $0.25 < x < 0.75$, and $(3.810\,91, 46.281\,6, -6.109\,78, 0.109\,960, 0.109\,961, 7.723\,00, 7.723\,00)$ for $x > 0.75$. Here, $B_x = 2$ and $\gamma = 5/3$. The initial condition is shown by the dashed lines in Figure 2 and represents two fast shocks propagating toward each other with Mach numbers 10 and 8. The profiles at $t = 0.035$ and 0.14 are shown by the dotted and solid lines, respectively, in Figure 2. The dotted lines demonstrate the correctness of the scheme for the propagation of the strong shocks. All the numerical results are in good agreement with those for the same problem by a high-order Godunov scheme developed by Dai and Woodward^[2].

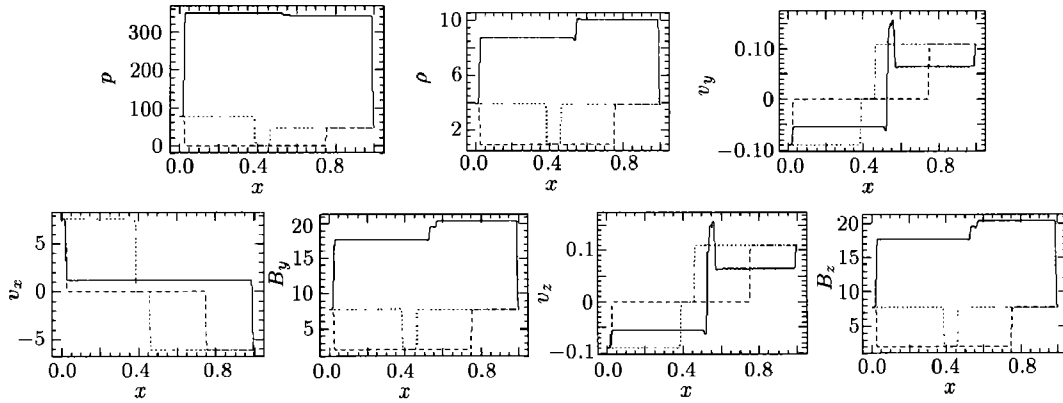


Fig.2 The profiles of various parameters for the collision of two fast shocks with Mach numbers 10 and 8: dashed lines: $t = 0$; solid lines: $t = 0.14$; dotted lines: $t = 0.035$

5 2-D Case: Orszag and Tang's MHD Vortex System

As an effort to apply the two-step macCormack scheme for multidimensional MHD problem, the Orszag and Tang's MHD vortex system^[28] is calculated here. This compressible flow proposed by Orszag and Tang contains significant features of MHD turbulence, and has taken as a standard numerical test. The flow involves complex evolution due to interactions between shock waves and the vortices. The initial conditions are as follows:

$$\begin{aligned} \rho(x, y, 0) &= \gamma^2, & v_x(x, y, 0) &= -\sin y, & v_y(x, y, 0) &= \sin x, & v_z(x, y, 0) &= 0, \\ p(x, y, 0) &= \gamma, & B_x(x, y, 0) &= -\sin y, & B_y(x, y, 0) &= \sin 2x, & B_z(x, y, 0) &= 0, \end{aligned}$$

with $\gamma = 5/3$.

The computational domain is $[0, 2\pi] \times [0, 2\pi]$ with periodic boundary condition in both x and y directions. We used a uniform mesh of 200×200 grid nodes. We have successfully run our calculation from $t = 0$ to $t = 8$. It should be noted that we employ a correction

procedure^{[18],[27]} based on solving a Poisson equation to enforce the divergence free condition of the magnetic field. This procedure maintained the integrity of the numerical solution and stabilized the calculation for long-term evolution of the MHD field. In fact, our program runs for much longer time and the calculation is stable and solution is reasonable.

For the purpose of comparison, here are shown the density contours at $t = 0.5$, $t = 2$ and $t = 3$ using the same contour level as that in Jiang and Wu^[27]. Figure 3 sees very complex shock structure. At $t = 2$, an intermediate shock is formed at the shock front in the region of $\pi < x < 1.5\pi$ and $0 < y < 0.75\pi$. At $t = 3$, the fast shock front in the region of $1.25\pi < x < 1.5\pi$ and $0.5\pi < y < 0.75\pi$, and a slow shock front in the region of $0.5\pi < x < \pi$ and $0.5\pi < y < 0.75\pi$. We found that our results are almost identical to Jiang and Wu's result^[27], obtained by using a WENO scheme.

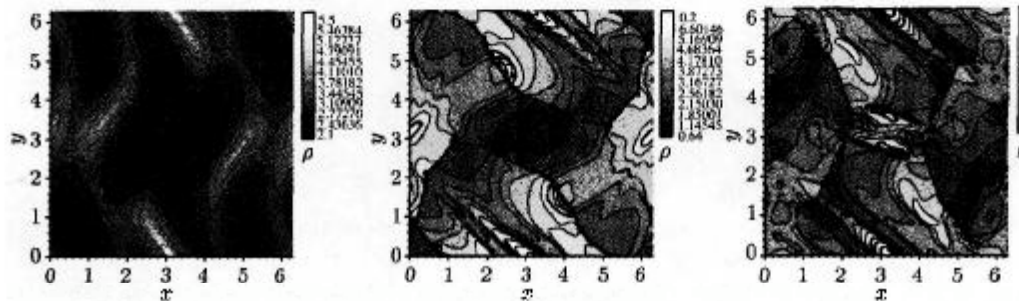


Fig.3 The density profiles of Orszag and Tang's two-dimensional MHD problem, left for density contours at $t = 0.5$, middle for density contours at $t = 2$, and right for density contours at $t = 3$

6 Conclusion

A new two-step TVD MacCormack type numerical scheme of second order accuracy is proposed for MHD equations. Different from other TVD type numerical schemes, our chosen scheme has been simplified by avoiding the splitting of Jacobian matrix and thus the calculation of eigenvectors. The numerical experiments of a self-similar Cauchy problem with the discontinuous initial data consisting of constant states, the collision of two fast MHD shocks, and two-dimensional Orszag and Tang's MHD vortex problem can give us reliable numerical results as compared to the former existing results and show the simplicity and practicability of the new scheme.

By the way, the application of this numerical scheme to the numerical study of solar wind is left for future work.

References

- [1] Dai W, Woodward P R. On the divergence-free condition and conservation laws in numerical simulations for supersonic magnetohydrodynamic flows. *Ap. J.*, 1998, 494:317—335
- [2] Dai Wentong, Woodward P R. A high-order Godunov-type scheme for shock interactions in ideal magnetohydrodynamics. *SIAM J. Sci. Comp.*, 1997, 18(1):957—981

- [3] Feng Xueshang *et al.* Numerical methods in solar wind simulations. *Space Sci. Rev.*, 2003, **107**:43—53
- [4] Feng Xueshang, Wu S T, Fan Quanli, Wei Fengsi. A class of TVD type combined numerical scheme for MHD equations and its application to MHD numerical simulation. *Chin. J. Space Sci.*, 2002, **22**(4):316—323
- [5] Jeffrey A, Taniuti T. Non-Linear Wave Propagation. New York: Academic Press, 1964
- [6] Kulikovskiy A G, Lyubimov G A. Magnetohydrodynamics. Reading: Addison Wesley, 1965
- [7] Garcia-Navarro P, Alerudo F, Saviron J M. 1-D open channel flow simulation using TVD-MacCormack scheme. *J. Hydraul. Eng.*, 1992, **118**(10):1 359—1 372
- [8] Jha A K, Akiyama J, Ura M. Modeling unsteady open-channel flows-modification to Beam and Warming scheme. *J. Hydraul. Eng.*, 1994, **120**(4):461—476
- [9] Jackson J D. Classical Electrodynamics, 3rd ed. New York: Wiley, 1999
- [10] Fan Quanlin, Wei Fengsi, Feng Xueshang. Start-time of magnetic reconnection in interplanetary space. *Chin. Phys. Lett.*, 2003, **20**(4):538—541
- [11] Yu Heng, Liu Yuping. A second-order accurate, component-wise TVD scheme for nonlinear, hyperbolic conservation laws. *J. Comp. Phys.*, 2001, **173**:1—16
- [12] Paul J Dellar. A note on magnetic monopoles and the one-dimensional MHD Riemann problem. *J. Comp. Phys.*, 2001, **172**:392—398
- [13] Janhunen P. A positive conservative method for magnetohydrodynamics based on HLL and Roe methods. *J. Comp. Phys.*, 2000, **160**:649—661
- [14] Kenneth G Powell, Philip L Roe, Timur J Linde *et al.* A solution-adaptive upwind scheme for ideal magnetohydrodynamics. *J. Comp. Phys.*, 1999, **154**:284—309
- [15] Brio M, Wu C C. An upwind differencing scheme for the equations of ideal magnetohydrodynamics. *J. Comp. Phys.*, 1998, **75**:400—422
- [16] Vincent S, Caltagirone J P. Numerical modelling of bore propagation and run-up on sloping beaches using MacCormack TVD scheme. *J. Hydraul. Res.*, 2001, **39**:41—49
- [17] Toth G, Odstrcil D. Comparison of some flux corrected transport and total variation diminishing numerical schemes for hydrodynamic and magnetohydrodynamic problems. *J. Comp. Phys.*, 1996, **128**:82—100
- [18] Toth G. The $\nabla \cdot \mathbf{B} = 0$ constraint in shock-capturing magnetohydrodynamics codes. *J. Comp. Phys.*, 2000, **161**:605—652
- [19] Roe P L, Balsara D S. Notes on the eigensystem of magnetohydrodynamics. *SIAM J. Appl. Math.*, 1996, **56**:57—67
- [20] Godunov S K. A finite difference method for the numerical computation of discontinuous solutions of the equations of fluid dynamics. *Mat. Sb.*, 1959, **47**:357—393
- [21] Sweby P K. High resolution schemes using flux limiters for hyperbolic conservation laws. *SIAM J. Num. Ana.*, 1984, **21**(5):995—1011
- [22] Harten A. High resolution schemes for hyperbolic conservation laws. *J. Comp. Phys.*, 1983, **49**:357—393
- [23] Fu Dexun, Ma Yanwen. Computational Fluid Dynamics. Beijing: High Education Press, 2002
- [24] Tan W Y. Computational Hydraulic Dynamics. Beijing: Tsinghua University Press, 1998
- [25] Yee H C. A class of high resolution explicit and implicit shock-capturing methods. NASA TM-101088, 1989
- [26] Sterck H D. Multi-dimensional upwind constrained transport on unstructured grids for “shallow water” magnetohydrodynamics. AIAA, 2001, 2001—2623
- [27] Jiang G S, Wu C C. A high-order WENO finite difference scheme for the equations of ideal magnetohydrodynamics. *J. Comp. Phys.*, 1999, **150**:561—594
- [28] Orszag S A, Tang C M. Small-scale structure of two-dimensional magnetohydrodynamic turbulence. *J. Fluid. Mech.*, 1979, **90**:129—143